MTH 508/508: Quiz 1 solutions

1. Consider the mapping $f : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$f(x_1, x_2, x_3) = (x_1^2 - x_2^2, x_1x_2, x_1x_3, x_2x_3).$$

Since $f(x) = f(-x)$, $f|_{S^2}$ defines a mapping $\bar{f} : \mathbb{R}P^2 \to \mathbb{R}^4$. Show that $\bar{f}$ is a smooth embedding.

Solution. Let $g = f|_{S^2}$ and $p : S^2 \to \mathbb{R}P^2$ be the covering map induced by the properly discontinuous $\mathbb{Z}_2$ action on S^2. By Corollary 1.3.2 (x) of the Lesson Plan, it follows that p is a C^∞ map. Clearly g is C^∞ since S^2 is a regular submanifold of $\mathbb{R}^3$ and each component of f is a polynomial. Since $f(x) = f(-x)$, f is constant on the fibers of p, and hence by the universal property of quotient topology (see Theorem 22.2 of Munkres), it follows that g induces a unique continuous map $\tilde{g} : \mathbb{R}P^2 \to \mathbb{R}^4$ such that $g = \tilde{g} \circ p$. Now, we note that $f(x) = f(y)$ if and only if either $x = y = 0$, or $y = \lambda x$ for some $\lambda \in \mathbb{R} \setminus \{0\}$ (why?). Hence $g(x) = g(y)$ if and only if $y = -x.$, which implies that $\tilde{g}$ is injective.

It remains to be shown that $\tilde{g}$ is an immersion. Since p is a local diffeomorphism and $\tilde{g}(x) = g(p^{-1}(x))$, it suffices to show that g is an immersion (why?). Thus, we need to show that the Jacobian Df of f given by:

$$Df = \begin{pmatrix} 2x_1 & -2x_2 & 0 \\ x_2 & x_1 & 0 \\ x_3 & 0 & x_1 \\ 0 & x_3 & x_2 \end{pmatrix},$$

has constant rank 2 that each point $p = (x_1, x_2, x_3) \in S^2$. First, we observe that the image $\mathrm{Im}(Df)$ is spanned by any two of the (tangent) vectors $X_p = (x_2, -x_1, 0)$, $Y_p = (x_3, 0, -x_1)$, and $Z_p = (0, x_3, -x_2)$ (why?). Finally, since at each $p \in S^2$, exactly two of the vectors $Df(X_p)$, $Df(Y_p)$, and $Df(Z_p)$ are linearly independent (why?), g has rank 2, and our assertion follows.

2. Let N be a discrete normal subgroup of a connected Lie group G. Show that $N \subset Z(G)$.

Solution. For any $n \in N$, we wish to show that $n \in Z(G)$. This is equivalent to showing that the set $S_h = \{g \in G : g^{-1}ng = n\}$ is both open and closed in G, for that would then imply that $S_n = G$ (since G is connected). Let $e \in G$ be the identity. Since N is discrete, there exists a neighborhood $U \ni e$ in G such that $U \cap N = \{e\}$. Thus for all $n \in N$, we have that $nU \cap N = \{n\}$. Now consider that map $\varphi_n : G \to N$ defined by $\varphi_n(g) = n^{-1}gn$. Note that map φ_n is continuous (why?). Consequently, since $S_n = \varphi^{-1}(nU)$ and $G \setminus S_n = \bigcup_{n' \in N \setminus \{n\}} \varphi^{-1}(n'U)$ (why?), S_n is open and closed. Thus, our assertion follows.